MODULE – III

AXISYMMETRIC PROBLEMS IN ELASTICITY

AXISYMMETRIC PROBLEMS

Equations in polar coordinates (2D) –

- Equilibrium equations,
- Strain-displacement relations,
- Airy's equation,
- Stress function and Stress components

Axisymmetric problems –

- Governing equations
- Application to thick cylinders
- Rotating discs

AXISYMMETRIC PROBLEMS

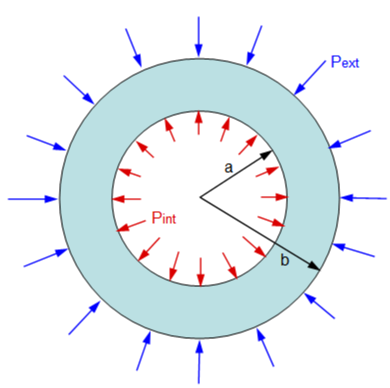
Axisymmetric Problems:

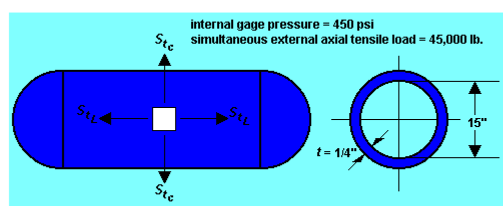
Solids of revolution deforms symmetrically with respect to the axis of revolution.

Eg:

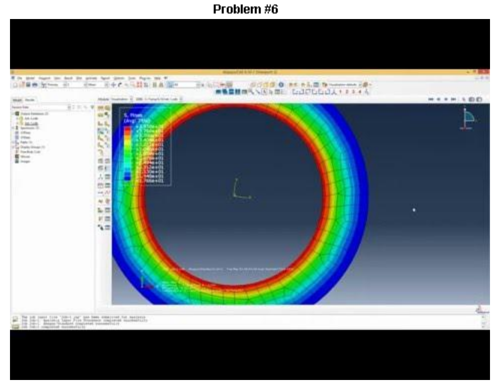
1. Cylinders subjected to internal and external pressures.
2. Rotating Circular Disks.

AXISYMMETRIC PROBLEMS

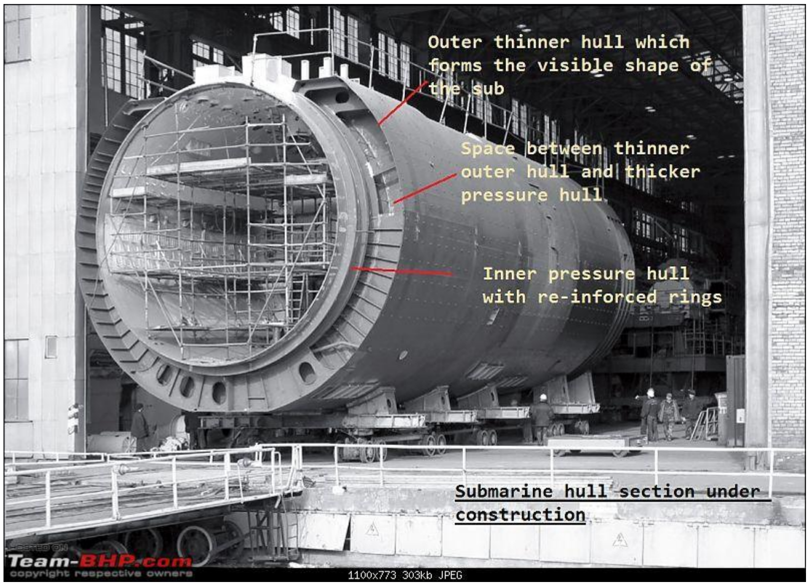




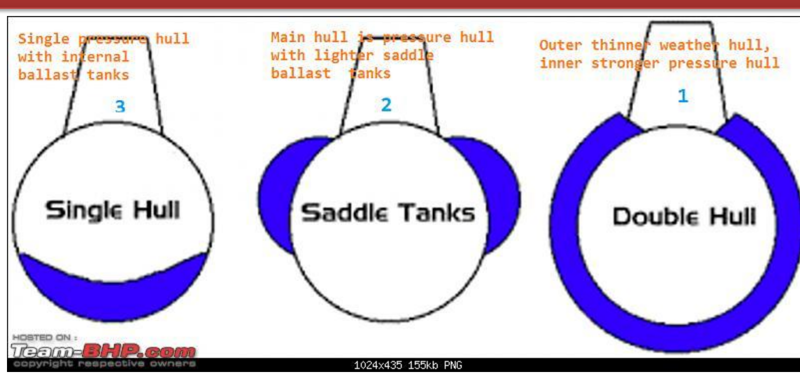
Problem #6



AXISYMMETRIC PROBLEMS



AXISYMMETRIC PROBLEMS



AXISYMMETRIC PROBLEMS

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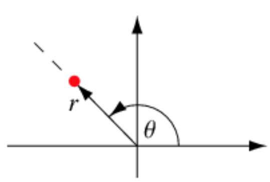
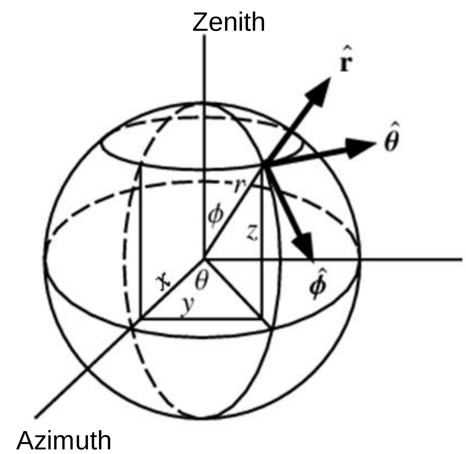
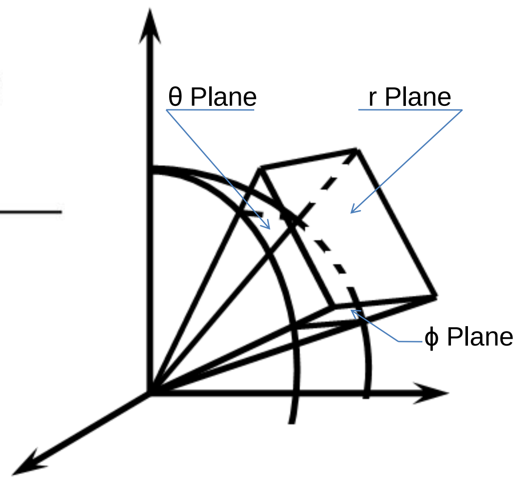
POLAR COORDINATE SYSTEM

In mathematics, the polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction.

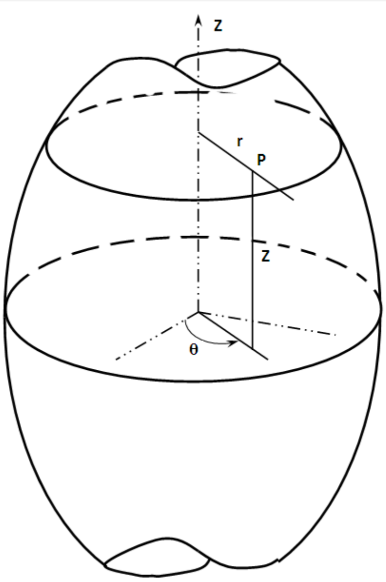


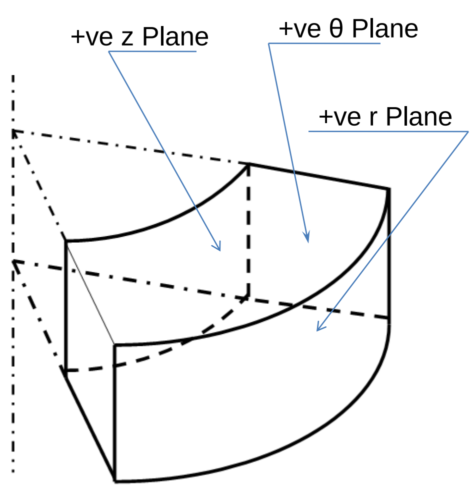
SPHERICAL COORDINATES





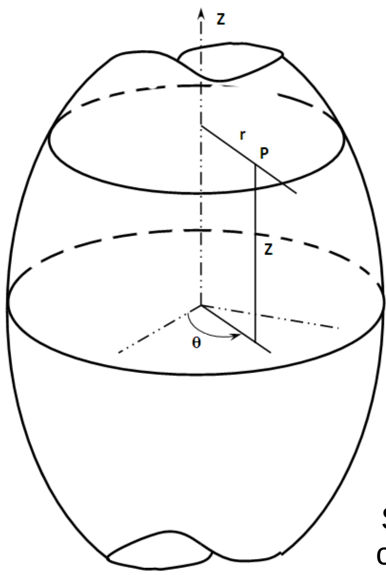
CYLINDRICAL COORDINATES

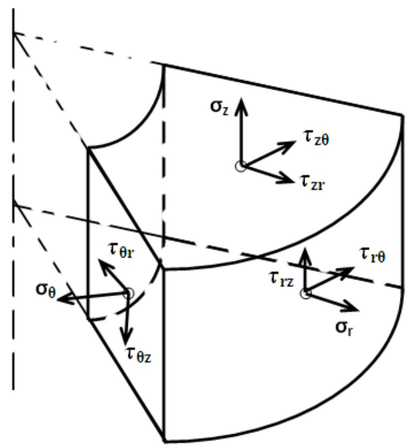




Positive Planes in Cylindrical Coordinates

AXISYMMETRIC PROBLEMS





Stresses components in cylindrical coordinates on a Cylinder Segment

EQUS. IN POLAR COORDINATES

Stress components in Cylindrical Coordinates are :

$\sigma_r, \sigma_z, \sigma_\theta, \tau_{rz}, \tau_{z\theta}, \tau_{r\theta}$

Differential Equations of Equilibrium in Cylindrical Co ordinates:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + \gamma_r = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\tau_{rz}}{r} + \gamma_z = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} + \gamma_\theta = 0$$

EQUS. IN POLAR COORDINATES

Diff Equ of Equil for Axisymmetric Problems:

Since the deformation is symmetrical stress components do not depend on θ and $\tau_{z\theta}$ & $\tau_{r\theta}$ do not exist

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + \gamma_r = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + \gamma_z = 0$$

EQUS. IN POLAR COORDINATES

In plane stress condition only the following stress components exist:

σ_r, σ_θ & $\tau_{r\theta}$

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + \gamma_r = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} + \gamma_\theta = 0$$

EQUS. IN POLAR COORDINATES

Strain Displacement Equ. in Cylindrical Coordinates

$$\varepsilon_r = \frac{\partial U_r}{\partial r}$$

$$\varepsilon_\theta = \frac{U_r}{r} + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta}$$

$$\varepsilon_z = \frac{\partial U_z}{\partial z}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial U_r}{\partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}$$

EQUS. IN POLAR COORDINATES

Strain Displacement Equ. for axisymmetric problems

$$\varepsilon_r = \frac{\partial U_r}{\partial r}$$

$$\varepsilon_\theta = \frac{U_r}{r}$$

$$\varepsilon_z = \frac{\partial U_z}{\partial z}$$

EQUS. IN POLAR COORDINATES

Constitutive Relations/Hooke's Law in Polar Coordinates:

$$\varepsilon_r = \frac{1}{E}\left[\sigma_r - \nu\left(\sigma_\theta + \sigma_z\right)\right]$$

$$\varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu\left(\sigma_r + \sigma_z\right)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_r + \sigma_\theta\right)\right]$$

EQUS. IN POLAR COORDINATES

Constitutive Relations/Hooke's Law for plane stress:

$$\varepsilon_r = \frac{1}{E}\left[\sigma_r - \nu\,\sigma_\theta\right]$$

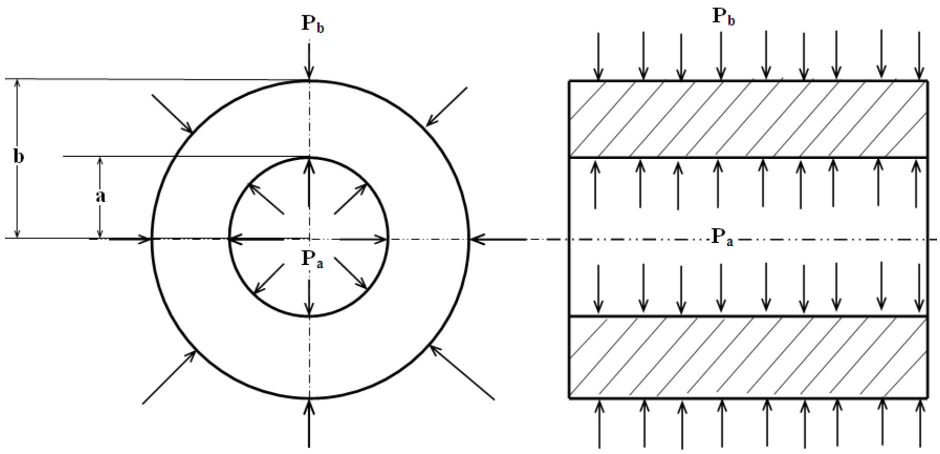
$$\varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu\,\sigma_r\right]$$

$$\varepsilon_z = -\frac{\nu}{E}\left[\left(\sigma_r + \sigma_\theta\right)\right]$$

THICK CYLINDERS

Thick cylinders subjected to internal and external pressure:-

(Lame's Problem)



THICK CYLINDERS

Thick cylinders subjected to internal and external pressure:-

(Lame's Problem) **Plane Stress:**

$$U_r = \left(\frac{1-\nu}{E}\right)\left[\frac{P_a a^2 - P_b b^2}{b^2 - a^2}\right] r + \left(\frac{1+\nu}{E}\right)\frac{a^2 b^2}{r}\left[\frac{P_a - P_b}{b^2 - a^2}\right]$$

$$\sigma_r = \left[\frac{P_a a^2 - P_b b^2}{b^2 - a^2}\right] - \frac{a^2 b^2}{r^2}\left[\frac{P_a - P_b}{b^2 - a^2}\right]$$

$$\sigma_\theta = \left[\frac{P_a a^2 - P_b b^2}{b^2 - a^2}\right] + \frac{a^2 b^2}{r^2}\left[\frac{P_a - P_b}{b^2 - a^2}\right]$$

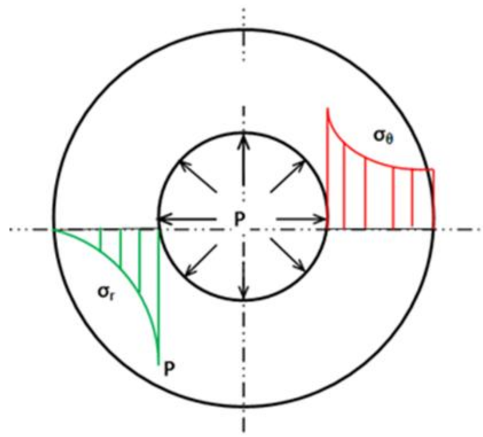
$$u_r = \frac{r}{E}(\sigma_\theta - \nu\sigma_r)$$

THICK CYLINDERS

Cylinder subjected to internal pressure P :-

$$\sigma_r = \frac{P\ a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$

$$\sigma_\theta = \frac{P\ a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)$$

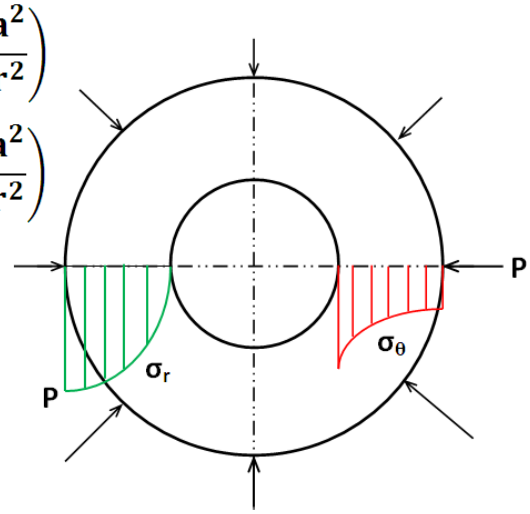


THICK CYLINDERS

Cylinder subjected to external pressure P :-

$$\sigma_r = -\frac{P\ b^2}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_\theta = -\frac{P\ b^2}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right)$$



THICK CYLINDERS

Thick cylinders subjected to internal and external pressure:-

(Lame's Problem)

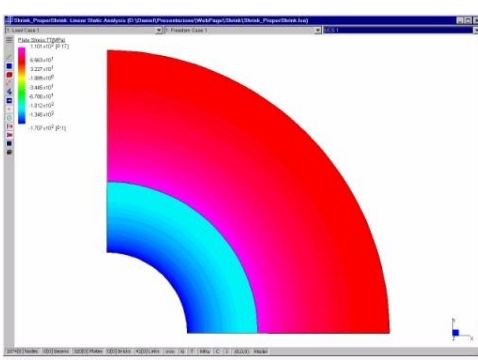
Plane Strain:

$$\sigma_r = \left[\frac{P_a a^2 - P_b b^2}{b^2 - a^2}\right] - \frac{a^2 b^2}{r^2}\left[\frac{P_a - P_b}{b^2 - a^2}\right]$$

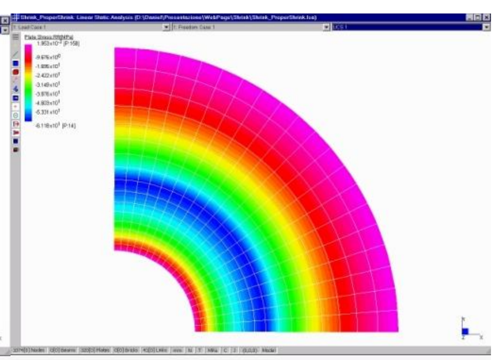
$$\sigma_\theta = \left[\frac{P_a a^2 - P_b b^2}{b^2 - a^2}\right] + \frac{a^2 b^2}{r^2}\left[\frac{P_a - P_b}{b^2 - a^2}\right]$$

$$\sigma_z = 2\nu\left(\frac{P_b a^2 - P_a b^2}{b^2 - a^2}\right)$$

AXISYMMETRIC PROBLEMS



Hoop stresses contour plot.



Radial Stresses contour plot.

Stress contour plot of Hoop Stress & Radial Stress for a thick cylinder subjected to internal pressure.

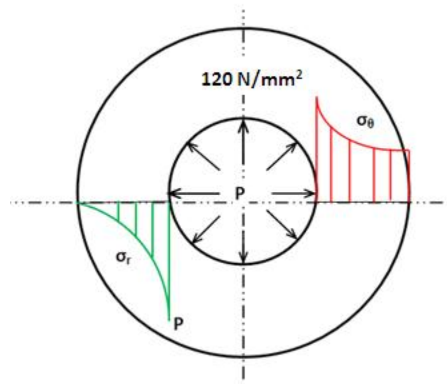
THICK CYLINDERS

A thick cylinder of internal diameter 160 mm is subjected to an internal pressure of 40 N/mm^2. If the allowable stress in the material is 120 N/mm^2, find the thickness required.

Ans: Thickness = 33.14 mm

$$\sigma_r = \frac{P\ a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$

$$\sigma_\theta = \frac{P\ a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)$$



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THICK CYLINDERS

A thick walled tube with an internal radius of 12 cm is subjected to an internal pressure of 200 Mpa (E = 2.1 x 10^5 Mpa and ν = 0.3). Determine the optimum value of external radius if the maximum shear stress developed is 350 MPa. Also determine the change in internal radius due to the pressure

Ans: b = 18.33cm; U_a = 0.032 cm.

$$\sigma_r = \frac{P\ a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$

$$\sigma_\theta = \frac{P\ a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)$$

$$U_r = \left(\frac{1 - \nu}{E}\right)\left[\frac{P_a a^2 - P_b b^2}{b^2 - a^2}\right] r + \left(\frac{1 + \nu}{E}\right)\frac{a^2 b^2}{r}\left[\frac{P_a - P_b}{b^2 - a^2}\right]$$

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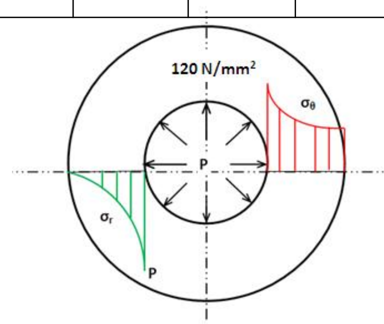
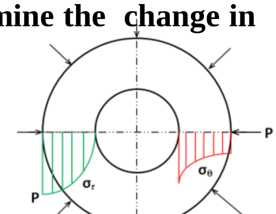
THICK CYLINDERS

r	120	130	140	150	160	170	180	183.3
σ_r	-200.00	-148.22	-107.14	-74.00	-46.87	-24.39	-5.55	0.00
σ_θ	500.02	448.24	407.16	374.01	346.89	324.41	305.57	300.02
$\tau_{r\theta}$	350.01	298.23	257.15	224.01	196.88	174.40	155.56	150.01



THICK CYLINDERS

A thick walled tube with an internal radius of 12 cm is subjected to an external pressure of 200 Mpa (E = 2.1 x 10^5 Mpa and ν = 0.3). Determine the optimum value of external radius if the maximum shear stress developed is 350 MPa. Also determine the change in internal radius due to the pressure



r	120	130	140	150	160	170	180	183.3
σ_r	0	-51.78	-92.86	-126	-153.1	-175.6	-194.4	-200
σ_θ	-700	-648.2	-607.2	-574	-546.9	-524.4	-505.6	-500.02
$\tau_{r\theta}$	350.01	298.23	257.15	224	196.9	174.4	155.6	150.01

THICK CYLINDERS

- The shear stress at any point on a cylinder subjected to internal and external pressure is given by:

$$\tau_{max} = \frac{\sigma_\theta - \sigma_r}{2} \qquad \tau_{max} = 35000\ N/cm^2$$

The stress distribution on a cylinder subjected to internal pressure shows that the shear stress will be maximum at the inner surface.

At the inner surface, r = a;

$$\sigma_r = -P = -200\ MPa = -20000\ N/cm^2$$

$$\sigma_\theta = \frac{P\,a^2}{b^2 - a^2}\left(1 + \frac{b^2}{a^2}\right)$$

THICK CYLINDERS

$$= P.\frac{b^2+a^2}{b^2-a^2} = 20000.\frac{b^2+12^2}{b^2-12^2}$$

$$2.\tau_{max} = 20000.\frac{b^2+12^2}{b^2-12^2} - -20000$$

$$2 \text{ x } 1.75 = \frac{b^2+12^2+b^2-12^2}{b^2-12^2}$$

$$\frac{b^2}{b^2-12^2} = 1.75$$

b = 18.33 cm. ***Ans***

THICK CYLINDERS

$$U_r = \left(\frac{1-\nu}{E}\right)\left[\frac{P_a a^2 - P_b b^2}{b^2 - a^2}\right] r + \left(\frac{1+\nu}{E}\right)\frac{a^2 b^2}{r}\left[\frac{P_a - P_b}{b^2 - a^2}\right]$$

$$U_r = \left(\frac{1-0.3}{2.1 \times 10^5}\right)\left[\frac{200 \times 12^2}{18.33^2 - 12^2}\right] 12 + \left(\frac{1+0.3}{2.1 \times 10^5}\right)\frac{12^2 \times 18.33^2}{12}\left[\frac{200}{18.33^2 - 12^2}\right]$$

U_r = 0.032 cm ***Ans***

THICK CYLINDERS

If the factor of safety is given use the following equation to get the permissible stress:

$$\text{Factor of Safety} = \frac{\text{yield stress}}{\text{Permissible Stress}}$$

Any of the failure theories can be used for the design:

Use, $\sigma_1 = \sigma_\theta;\ \sigma_2 = 0;\ \sigma_3 = \sigma_r$

THEORIES OF FAILURE

THEORIES OF FAILURE:

Failure depends on mode of failure i.e., ductile or brittle and the factor such as stress, strain and energy.

- σ_y is the yield stress for the material in a uniaxial test.
- $\sigma_{1,}$ σ_2 and σ_3 are the principal stresses such that $\sigma_1 > \sigma_2 > \sigma_3$

1. **Maximum principal stress theory:**

According to maximum principal stress theory, failure occurs when $\sigma_1 > \sigma_y$.

THEORIES OF FAILURE

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- σ_y is the yield stress for the material in a uniaxial test.
- $\sigma_{1,}$ σ_2 and σ_3 are the principal stresses such that $\sigma_1 > \sigma_2 > \sigma_3$

2. **Maximum Shearing Stress Theory:**

According to maximum shearing stress theory, failure occurs when $\frac{\sigma_1 - \sigma_3}{2} \geq \frac{\sigma_y}{2}$

THEORIES OF FAILURE

THEORIES OF FAILURE:

- σ_y is the yield stress for the material in a uniaxial test.
- $\sigma_{1,}$ σ_2 and σ_3 are the principal stresses such that $\sigma_1 > \sigma_2 > \sigma_3$

3. Maximum Elastic Strain Theory:

According to maximum Elastic Strain theory, failure occurs when

$$\frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)] \geq \frac{\sigma_y}{E}$$

THEORIES OF FAILURE

THEORIES OF FAILURE:

- σ_y is the yield stress for the material in a uniaxial test.
- $\sigma_{1,}$ σ_2 and σ_3 are the principal stresses such that $\sigma_1 > \sigma_2 > \sigma_3$

4. Octahedral Shearing Stress Theory:

According to maximum Octahedral Shearing Stress theory, failure occurs when

$$\frac{1}{3}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2} \geq \frac{\sqrt{2}}{3}\sigma_y$$

THEORIES OF FAILURE

THEORIES OF FAILURE:

- **σ_y is the yield stress for the material in a uniaxial test.**
- **σ_1, σ_2 and σ_3 are the principal stresses such that $\sigma_1 > \sigma_2 > \sigma_3$**

5. **Maximum elastic energy Theory:**

According to maximum elastic energy theory, failure occurs when

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \geq \sigma_y^2$$

THEORIES OF FAILURE

THEORIES OF FAILURE:

- **σ_y is the yield stress for the material in a uniaxial test.**
- **σ_1, σ_2 and σ_3 are the principal stresses such that $\sigma_1 > \sigma_2 > \sigma_3$**

6. **Energy of distortion theory:**

According to maximum Energy of distortion theory, failure occurs when

$$\frac{1}{3}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2} \geq \frac{\sqrt{2}}{3}\sigma_y$$

*** This identical to the octahedral shearing stress theory.**

COMPOSITE TUBES

STRESSES IN COMPOSITE TUBES – INTERFERENCE FIT

COMPOSITE TUBES

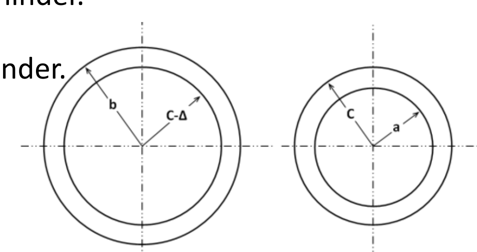
In a shrink fitted composite tube two cylinders of different material or same material is fitted one inside another.

a – Inner radius of the inner cylinder.

c – Outer radius of the inner cylinder.

c-Δ – Inner radius of the outer cylinder.

b – Outer radius of the outer cylinder.



COMPOSITE TUBES

The two cylinders are assembled by heating the outer cylinder and cooling the inner cylinder.

The composite tubes can with stand very high pressure of the order of 15000 bar.

If we need a normal tube to withstand such a high pressure the yield stress of the material must be at least 2940 MPa. Since no such high-strength material exist, shrink fitted composite tubes are designed.

COMPOSITE TUBES

P_c is the contact pressure due to shrink fit.

The contact pressure acts on the outer surface of the inner cylinder and inner surface of the outer cylinder.

u_1 – Reduction in outer radius of the inner cylinder due to contact pressure P_c.

u_2 – Increase in inner radius of the outer cylinder due to contact pressure P_c.

$$-U_1 + U_2 = \Delta$$

COMPOSITE TUBES

Substituting the expression for U_1 and U_2 in the above expression we get,

$$P_C = \frac{\Delta / c}{\frac{1}{E_1}\left[\frac{c^2 + a^2}{c^2 - a^2} - \nu_1\right] + \frac{1}{E_2}\left[\frac{b^2 + c^2}{b^2 - c^2} + \nu_2\right]}$$

The above expression give the contact pressure, P_c due to shrink fit.

If the two cylinders are made of the same material.

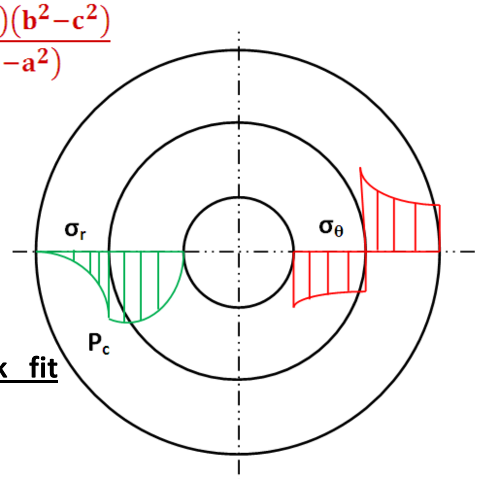
Then, $E_1 = E_2$; $\nu_1 = \nu_2$

COMPOSITE TUBES

If the two cylinders are made of the same material.

Then, $E_1 = E_2$; $\nu_1 = \nu_2$

$$P_C = \frac{E\Delta}{2c^3} \frac{(c^2 - a^2)(b^2 - c^2)}{(b^2 - a^2)}$$



<u>Stress distribution in a shrink fit cylinder due to contact pressure.</u>

COMPOSITE TUBES

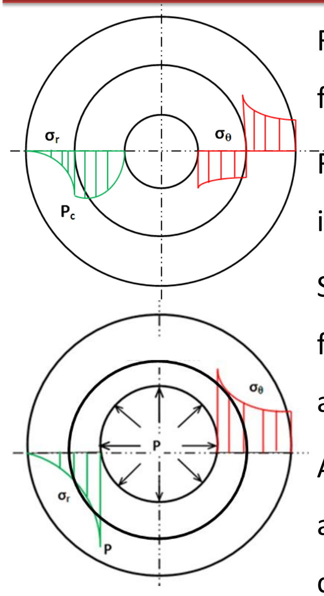


Fig a shows the stress distribution on the shrink fit due to the contact pressure.

Fig b shows the stress distribution due to internal pressure.

Sum of the stresses at any point on the shrink fit tube will give the net stress due to shrink fit and internal pressure.

At the inner surface of the inner tube p causes a tensile circumferential stress but the p_c causes a compressive circumferential stress.

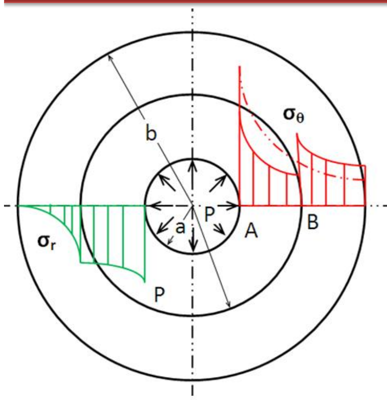
COMPOSITE TUBES

So the net stress at the inner surface of the inner wall will be less than the stress due to internal pressure alone.

Hence a composite cylinder can support greater internal pressure than an ordinary cylinder.

At the inner point of the external cylinder both the stress due to p and that due to p_c are tensile and they get added up.

COMPOSITE TUBES



For design purposes the shrink fit allowance can be chosen such that the two cylinders will have equal strength.

According to maximum shear stress theory:

$$(\sigma_1 - \sigma_3)_A = (\sigma_1 - \sigma_3)_B$$

Shrink Fit allowance required for getting equal strength is given by

$$\Delta = \frac{2P}{E} \cdot \frac{b^2 c(c^2 - a^2)}{b^2(c^2 - a^2) - c^2(b^2 - c^2)}$$

COMPOSITE TUBES

$$\sigma_\theta - \sigma_r = P\frac{2b^2}{b^2 - a^2}\left[1 - \frac{1}{\frac{b^2}{b^2 - c^2} + \frac{c^2}{c^2 - a^2}}\right]$$

The above quantity will be minimum when

$\frac{b^2}{b^2 - c^2} + \frac{c^2}{c^2 - a^2}$ is minimum

For a given values of P, a and b, the optimum values of c and Δ for which the value of $\sigma_\theta - \sigma_r$ is a minimum is given by:

$$c = \sqrt{ab} \text{ and } \Delta_{opt} = \frac{P}{E}\sqrt{ab}$$

COMPOSITE TUBES

A tube 96 mm in diameter is used to reinforce a tube 48 mm internal diameter and 72 mm outer diameter. The compound tube is made to with stand an internal pressure of 60 MPa. The shrinkage allowance is such that the final maximum stress in each tube is the same. Determine this stress and plot a diagram to show the variation of hoop stress in the two tubes. Also calculate the shrinkage allowance required.

COMPOSITE TUBES

Hoop stress = Circumferential stress = tangential stress.

Find the Hoop stress in terms of contact pressure at

Inner Cylinder: At r = 24 mm and r = 36 mm ($-3.6P_c$ and $-2.6\ P_c$)

Outer cylinder: At r = 36 mm and r = 48 mm ($3.572\ P_c$ and $2.572\ P_c$)

Consider the composite tube as a single unit and find the Hoop stress at r = 24mm, 30mm and 48 mm (100 Mpa, 55.6 Mpa and 40 Mpa)

Find the net stress at the inner and outer radii of both tubes.

COMPOSITE TUBES

Equate Maximum stress in the inner tube to maximum stress in the outer tube and find the contact pressure (6.19 MPa).

Δ = 0.0066 mm.

ROTATING DISCS

STRESSES IN ROTATING DISCS

ROTATING DISCS

STRESSES IN SOLID ROTATING DISC:

The stress distribution in rotating circular disks which are thin is given by:

b – Outer radius of the disk

ρ – Density of disk material.

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2(b^2 - r^2)$$

$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2 b^2 - \frac{1+3\nu}{8}\rho\omega^2 r^2$$

$$u_r = \frac{r}{E}(\sigma_\theta - \nu\sigma_r)$$

The stresses attain their maximum value at the centre of the disc.

(i.e., at r=0). $(\sigma_r)_{max} = (\sigma_\theta)_{max} = \frac{3+\nu}{8}\rho\omega^2 b^2$

ROTATING DISCS

STRESSES IN ROTATING DISC WITH A HOLE OF RADIUS a:

The stress distribution in rotating circular disk with a hole is given by:

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(b^2 + a^2 - \frac{a^2b^2}{r^2} - r^2\right)$$

$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2\left(b^2 + a^2 + \frac{a^2b^2}{r^2} - \frac{1+3\nu}{3+\nu}r^2\right)$$

$$u_r = \frac{r}{E}(\sigma_\theta - \nu\sigma_r)$$

b – Outer radius of the disk

ρ – Density of disk material.

a- radius of the hole

ω – Angular velocity in rad/s.

$$(\sigma_r)_{max} = \frac{3+\nu}{8}\rho\omega^2(b-a)^2 \text{ at } r = \sqrt{ab}$$

$$(\sigma_\theta)_{max} = \frac{3+\nu}{8}\rho\omega^2\left(b^2 + \frac{1-\nu}{3+\nu}a^2\right) \text{ at } r = a$$

ROTATING DISCS

A thin disc of uniform thickness is of 800 mm outer diameter and 50 mm inner diameter. It rotates at 3000 rpm. Determine the radial and the hoop stresses at radii of 0.25 mm, 50 mm, 100 mm, 150 mm, 200 mm, 300 mm and 400 mm. Density of the material is 7800 kg/m^3 , ν = 0.25, What are the maximum values of the radial, hoop and shear stresses.

(use SI units)

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(b^2 + a^2 - \frac{a^2b^2}{r^2} - r^2\right)$$

$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2\left(b^2 + a^2 + \frac{a^2b^2}{r^2} - \frac{1+3\nu}{3+\nu}r^2\right)$$

ROTATING DISCS

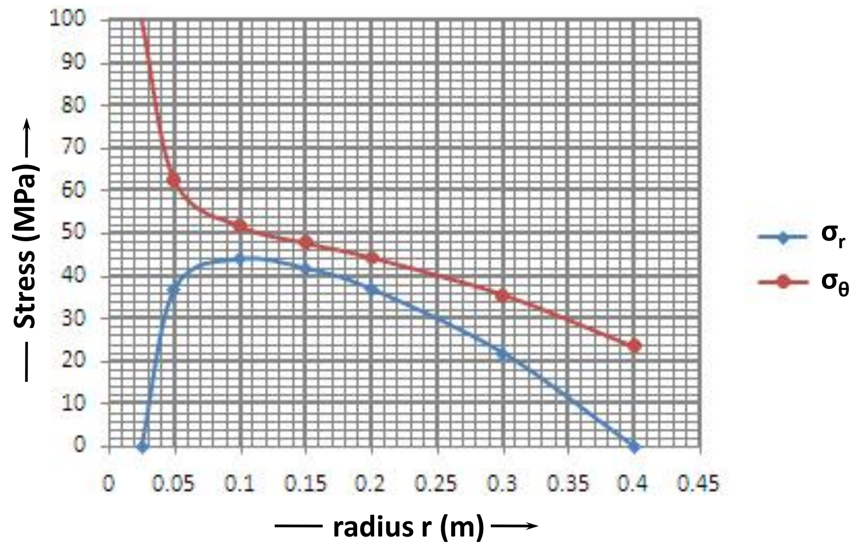
$$\text{Radial Stress, } \sigma_r = 312.75\left(0.1606 - \frac{0.0001}{r^2} - r^2\right)\text{ MPa}$$

r(m)	0.025	0.05	0.1	0.15	0.2	0.3	0.4
σ_r (Mpa)	0	36.94	43.97	41.8	36.94	21.73	0

Hoop Stress,

r(m)	0.025	0.05	0.1	0.15	0.2	0.3	0.4
σ_θ (Mpa)	100.17	62.32	51.68	47.83	44.28	35.423	23.48

ROTATING DISCS

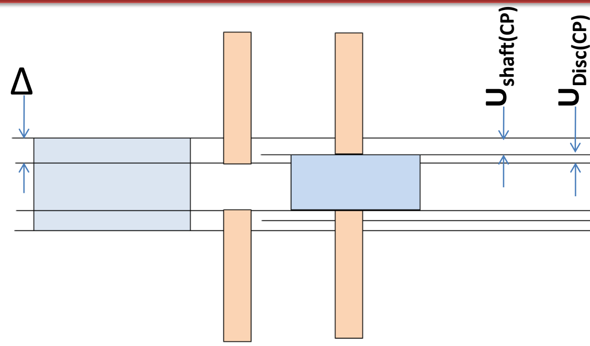


Note: At r = a, σ_r = 0

ROTATING DISCS

A hollow steel disc of 400 mm outer diameter and 100 mm inside diameter is shrunk fit on a steel shaft. The pressure between the disc and the shaft is 60 MPa. Determine the speed of the disc at which it loosen from the shaft neglecting the change in dimensions of the shaft due to rotation. $\rho = 7700$ kg/m^3 and $\nu = 0.3$.

ROTATING DISCS



Cylinder Subjected to external pressure (Shaft) a = 0; r = b =0.05; P = 60 x 10⁶

$$\sigma_r = -\frac{P\ b^2}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_\theta = -\frac{P\,b^2}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right)$$

Cylinder Subjected to internal pressure (disc) a = 0.05; r = b =0.2; P = 60 x 10⁶

$$\sigma_\theta = \frac{P\,a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)$$

$$\sigma_r = \frac{P\ a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$

$$u_r = \frac{r}{E}(\sigma_\theta - \nu\sigma_r)$$

$U_{disc\ (CP)} = -10^{-5}$m; $U_{disc\ (CP)} = 2.05 \times 10^{-5}$m

$\Delta = U_{disc\ (CP)} - U_{shaft\ (CP)}$

ROTATING DISCS

$\Delta = U_{disc\ (CP)} - U_{shaft\ (CP)} = 3.05 \times 10^{-5}$

$\Delta = U_{disc\ (rot)} - U_{shaft\ (rot)}$

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(b^2 + a^2 - \frac{a^2b^2}{r^2} - r^2\right)$$

$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2\left(b^2 + a^2 + \frac{a^2b^2}{r^2} - \frac{1+3\nu}{3+\nu}r^2\right)$$

$$u_r = \frac{r}{E}(\sigma_\theta - \nu\sigma_r)$$

Radial displacement of disc due to rotation:

a = 0.05; b = 0.2; r = 0.05; ρ = 7700;

$U_{disc(rot)} = 6.1302 \times 10\text{-}11\ \omega^2$ m

Radial displacement of shaft due to rotation:

a = 0; b = 0.05; r = 0.05; ρ = 7700;

$U_{shaft(rot)} = 8.021 \times 10\text{-}13\ \omega^2$ m

ω= 710 rad/sec; N = 6781.

ROTATING DISCS

Ur = 1.00 x 10^{-5} m.

When the disc is rotating , find the value of σ_θ and σ_r in terms of ω using equ. $$\sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(b^2 + a^2 - \frac{a^2b^2}{r^2} - r^2\right)$$

$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2\left(b^2 + a^2 + \frac{a^2b^2}{r^2} - \frac{1+3\nu}{3+\nu}r^2\right)$$

and thereby find the radial displacement using equ.

$$u_r = \frac{r}{E}(\sigma_\theta - \nu\sigma_r)$$

The disc will get loosened when this radial displacement is equal to 1.952 x 10^{-5} m

ROTATING DISCS

ω = 710.1 rad/s

rpm = 6781

ROTATING DISCS

A flat steel disc of 75 cm outside diameter with a 15 cm dameter hole is shrunk around a solid shaft. The shrink fit allowance is 1 part in 1000 (0.0075 cm in radius). E = 2.14 x 10^5 MPa.

At what rpm will the shrink fit loosen up as a result of rotation?

What is the circumferential stress in the disc when spinning at the above speed?

Assume that the same equations as for the disk are applicable to the solid rotating shaft also.

(use SI units)

ROTATING DISCS

ω = 475 rad/s

rpm = 4536

STRESS FUNCTION IN POLAR COORDINATES

Airy s Stress Function (ϕ) in polar coordinates can be defined as :

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \emptyset}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \emptyset}{\partial \theta^2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \emptyset}{\partial r^2}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \emptyset}{\partial \theta}\right)$$

STRESS FUNCTION IN POLAR COORDINATES

Stress Compatibility Equations:

Plane Stress:

$$\nabla^2(\nabla^2 \emptyset) = -(1+\nu)\left(\frac{\partial B_r}{\partial r} + \frac{B_r}{r} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta}\right)$$

Plane Strain:

$$\nabla^2(\nabla^2 \emptyset) = -\frac{1}{(1-\nu)}\left(\frac{\partial B_r}{\partial r} + \frac{B_r}{r} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta}\right)$$

$$\nabla^2 \emptyset = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \emptyset}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 \emptyset}{\partial \theta^2}\right)$$

In the absence of body forces: $\nabla^2(\nabla^2 \emptyset) = 0$

The above equation is also called ***Biharmonic equation***

STRESS FUNCTION IN POLAR COORDINATES

Show that the function $\emptyset = \left(Ar^2 + \frac{C}{r^2} + D\right) Cos2\theta$ satisfies the stress compatibility equation in polar coordinates in the absence of body forces. Find the components of stress.

In the absence of body forces $\nabla^2(\nabla^2\emptyset) = 0$

$$\nabla^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\emptyset}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2\emptyset}{\partial\theta^2}\right)\right) = 0$$

STRESS FUNCTION IN POLAR COORDINATES

$$\nabla^2\emptyset = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\emptyset}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2\emptyset}{\partial\theta^2}\right)$$

$$\frac{\partial\emptyset}{\partial r} = \left(2Ar - 2\frac{C}{r^3}\right) Cos2\theta$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial\emptyset}{\partial r}\right) = \frac{\partial}{\partial r}\left(2Ar^2 - 2\frac{C}{r^2}\right) Cos2\theta$$

$$= \left(4Ar + 4\frac{C}{r^3}\right) Cos2\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\emptyset}{\partial r}\right) = \left(4A + 4\frac{C}{r^4}\right) Cos2\theta$$

STRESS FUNCTION IN POLAR COORDINATES

$$\frac{\partial^2 \emptyset}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(-2\left(\mathrm{Ar}^2 + \frac{\mathrm{C}}{\mathrm{r}^2} + \mathrm{D}\right)\mathrm{Sin}2\theta\right)$$

$$\frac{\partial^2 \emptyset}{\partial \theta^2} = -4\left(\mathrm{Ar}^2 + \frac{\mathrm{C}}{\mathrm{r}^2} + \mathrm{D}\right)\mathrm{Cos}2\theta$$

$$\frac{1}{\mathrm{r}^2}\frac{\partial^2 \emptyset}{\partial \theta^2} = -4\left(\mathrm{A} + \frac{\mathrm{C}}{\mathrm{r}^4} + \frac{\mathrm{D}}{\mathrm{r}^2}\right)\mathrm{Cos}2\theta$$

$$\nabla^2 \emptyset = \frac{-4\mathrm{D}}{\mathrm{r}^2}\mathrm{Cos}2\theta$$

$$\nabla^2(\nabla^2 \emptyset) = \nabla^2\left(\frac{-4\mathrm{D}}{\mathrm{r}^2}\mathrm{Cos}2\theta\right)$$

STRESS FUNCTION IN POLAR COORDINATES

$$\nabla^2(\nabla^2 \emptyset) = \nabla^2\left(\frac{-4\mathrm{D}}{\mathrm{r}^2}\mathrm{Cos}2\theta\right)$$

$$= \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial\left(\frac{-4\mathrm{D}}{\mathrm{r}^2}\mathrm{Cos}2\theta\right)}{\partial \mathrm{r}}\right) + \frac{1}{\mathrm{r}^2}\left(\frac{\partial^2\left(\frac{-4\mathrm{D}}{\mathrm{r}^2}\mathrm{Cos}2\theta\right)}{\partial \theta^2}\right)$$

$$\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial\left(\frac{-4\mathrm{D}}{\mathrm{r}^2}\mathrm{Cos}2\theta\right)}{\partial \mathrm{r}}\right) = \frac{-16\mathrm{D}}{\mathrm{r}^4}\mathrm{Cos}2\theta$$

$$\frac{1}{\mathrm{r}^2}\left(\frac{\partial^2\left(\frac{-4\mathrm{D}}{\mathrm{r}^2}\mathrm{Cos}2\theta\right)}{\partial \theta^2}\right) = \frac{16\mathrm{D}}{\mathrm{r}^4}\mathrm{Cos}2\theta$$

STRESS FUNCTION IN POLAR COORDINATES

$$\nabla^2(\nabla^2\phi) = \frac{-16D}{r^4}\text{Cos}2\theta \quad + \frac{16D}{r^4}\text{Cos}2\theta$$

$$\nabla^2(\nabla^2\phi) = 0$$

$$\phi = \left(Ar^2 + \frac{C}{r^2} + D\right)\ \text{Cos}2\theta$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\frac{1}{r}\frac{\partial\phi}{\partial r} = \left(2A - 2\frac{C}{r^4}\right)\text{Cos}2\theta$$

$$\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = -4\left(A + \frac{C}{r^4} + \frac{D}{r^2}\right)\text{Cos}2\theta$$

$$\sigma_{rr} = -\left(2A + \frac{6C}{r^4} + \frac{4D}{r^2}\right)\text{Cos}2\theta$$

STRESS FUNCTION IN POLAR COORDINATES

$$\phi = \left(Ar^2 + \frac{C}{r^2} + D\right)\ \text{Cos}2\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}$$

$$\frac{\partial\phi}{\partial r} = \left(2Ar - 2\frac{C}{r^3}\right)\text{Cos}2\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = \left(2A + \frac{6C}{r^4}\right)\text{Cos}2\theta$$

STRESS FUNCTION IN POLAR COORDINATES

$$\emptyset = \left(Ar^2 + \frac{C}{r^2} + D\right) \text{Cos}2\theta$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\emptyset}{\partial\theta}\right)$$

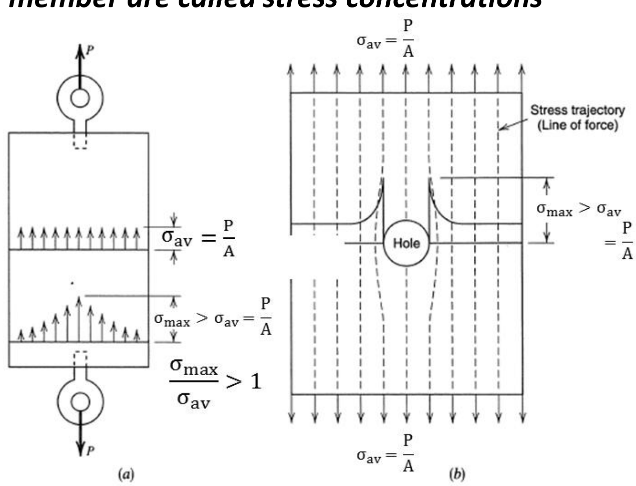
$$\frac{\partial\emptyset}{\partial\theta} = -2\left(Ar^2 + \frac{C}{r^2} + D\right) \text{Sin}2\theta$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(-2\left(Ar + \frac{C}{r^3} + \frac{D}{r}\right) \text{Sin}2\theta\right)$$

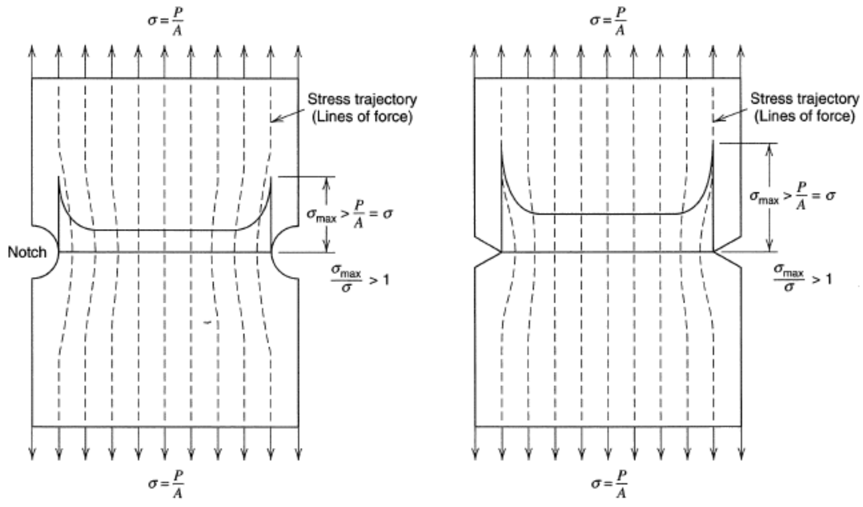
$$\tau_{r\theta} = \left(2A - \frac{6C}{r^4} - \frac{2D}{r^2}\right) \text{Sin}2\theta$$

STRESS CONCENTRATION

Large stresses resulting from discontinuities developed in a small portion of a member are called stress concentrations



STRESS CONCENTRATION



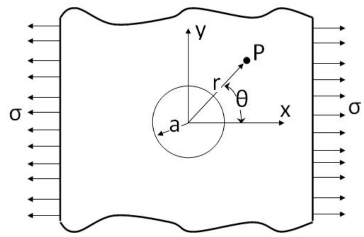
STRESS CONCENTRATION

Conditions for Stress concentration:

1. Abrupt changes in section eg: root of the thread of a bolt, at the bottom of a tooth on a gear, at a section of a plate or beam containing a hole, corner of a keyway in a shaft.
2. Contact Stresses at the point of application of the external forces – eg: at points of contact between gear teeth.
3. Discontinuities in material: eg: non metallic inclusions in steel.
4. Initial Stresses in a member – eg: residual stresses in welding.
5. Crack that exists in the member

STRESS FUNCTION IN POLAR COORDINATES

Stress concentration problem of a small hole in a large plate :



$$\sigma_{rr} = \frac{\sigma}{2}\left(1-\frac{a^2}{r^2}\right) + \frac{\sigma}{2}\left(1-\frac{a^2}{r^2}\right)\left(1-\frac{3a^2}{r^2}\right)\text{Cos}2\theta$$

$$\sigma_{\theta\theta} = \frac{\sigma}{2}\left(1+\frac{a^2}{r^2}\right) - \frac{\sigma}{2}\left(1+\frac{3a^4}{r^4}\right)\text{Cos}2\theta$$

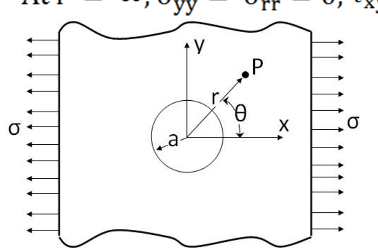
$$\tau_{r\theta} = -\frac{\sigma}{2}\left(1-\frac{a^2}{r^2}\right)\left(1+\frac{3a^2}{r^2}\right)\text{Sin}2\theta$$

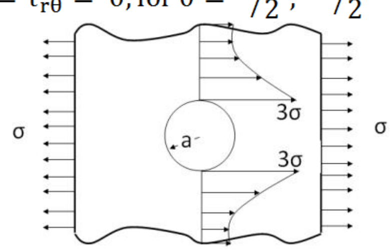
STRESS FUNCTION IN POLAR COORDINATES

At $r = a; \sigma_{rr} = \tau_{r\theta} = 0$, for all θ

At $r = \infty; \sigma_{xx} = \sigma_{rr} = \sigma; \quad \tau_{xy} = \tau_{r\theta} = 0$, for $\theta = 0, \pi$

At $r = \infty; \sigma_{yy} = \sigma_{rr} = 0; \tau_{xy} = \tau_{r\theta} = 0$, for $\theta = {}^{\pi}/_{2}, {}^{3\pi}/_{2}$





At $r = a; \sigma_{\theta\theta} = \sigma(1 - 2\text{Cos}2\theta)$

$$(\sigma_{\theta\theta})_{max} = 3\sigma, \text{for } \theta = {}^{\pi}/_{2}, {}^{3\pi}/_{2}$$

STRESS FUNCTION IN POLAR COORDINATES

Application of stress function to Lame's problem:

$$\emptyset(r) = A\log r + Br^2\log r + Cr^2 + D$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\emptyset}{\partial r} + \frac{1}{r^2}\frac{\partial^2\emptyset}{\partial\theta^2}$$

$$\frac{\partial\emptyset}{\partial\theta} = 0$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\emptyset}{\partial r} \quad \text{and} \quad \sigma_{\theta\theta} = \frac{\partial^2\emptyset}{\partial r^2}$$

$$\sigma_{rr} = \frac{1}{r}\frac{d\emptyset}{dr} = \frac{1}{r}\frac{d(A\log r + Br^2\log r + Cr^2 + D)}{dr}$$

$$= \frac{1}{r}\left[\frac{A}{r} + 2Br\log r + \frac{Br^2}{r} + 2Cr\right]$$

$$= \frac{A}{r^2} + B(1 + 2\log r) + 2C$$

STRESS FUNCTION IN POLAR COORDINATES

$$\sigma_{\theta\theta} = \frac{d^2\emptyset}{dr^2} = \frac{d\left[\frac{A}{r} + 2Br\log r + \frac{Br^2}{r} + 2Cr\right]}{dr}$$

$$= \frac{-A}{r^2} + 2B\log r + 2B + B + 2C$$

$$= \frac{-A}{r^2} + B(3 + 2\log r) + 2C$$

The boundary conditions can be applied as follows:

- Stress components varying along the radial direction
- Plane Stress as well as plane Strain Condition.
- Coefficient B must be zero from the consideration of displacement of thick cylinders.

STRESS FUNCTION IN POLAR COORDINATES

With B = 0, the stress function and components can be written as:

$$\emptyset(r) = A\log r + Br^2\log r + Cr^2 + D$$

$$\sigma_{rr} = \frac{A}{r^2} + 2C$$

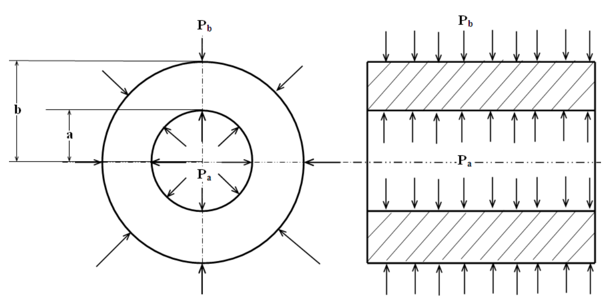
$$\sigma_{\theta\theta} = \frac{-A}{r^2} + 2C$$

$$\sigma_{rr}(r = a) = -P_a;\ \frac{A}{a^2} + 2C = -P_a$$

$$\sigma_{rr}(r = b) = -P_b;\ \frac{A}{b^2} + 2C = -P_b$$

$$A = \frac{(P_b - P_a)a^2b^2}{(b^2 - a^2)} \qquad 2C = \frac{P_a a^2 - P_b b^2}{(b^2 - a^2)}$$

STRESS FUNCTION IN POLAR COORDINATES



$$\sigma_r = \left[\frac{P_a a^2 - P_b b^2}{b^2 - a^2}\right] - \frac{a^2b^2}{r^2}\left[\frac{P_a - P_b}{b^2 - a^2}\right]$$

$$\sigma_\theta = \left[\frac{P_a a^2 - P_b b^2}{b^2 - a^2}\right] + \frac{a^2b^2}{r^2}\left[\frac{P_a - P_b}{b^2 - a^2}\right]$$

STRESS FUNCTION IN POLAR COORDINATES

Shear Centre:

- The transverse force applied at shear center does not lead to the torsion of thin-walled beam.
- The shear center is a center of rotation for a section of thin-walled beam subjected to pure torsion.
- The shear center is a position of shear flows resultant force, if the thin-walled beam is subjected to pure shear.